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A new method for proving chromatic uniqueness of graphs

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Abstract

We give a brief survey on a new method for proving the chromatic uniqueness of graphs by their adjoint polynomials. We obtain some simpler proofs of relevant theorems and a new family of chromatically unique graphs.

1. Introduction

The graphs that we consider here are finite, undirected, simple, and loopless. The reader may refer to [1] for all notation and terminology not explained here. Let $P(G,\lambda)$ denote the chromatic polynomial of a graph. Two graphs G and H are chromatically equivalent if $P(G,\lambda)=P(H,\lambda)$. A graph G is chromatically unique if $P(H,\lambda)=P(G,\lambda)$ implies that H is isomorphic to G, denoted by $H\cong G$. Since the appearance of the first paper on chromatically unique graphs by Chao and Whitehead [2] in 1978, many families of such graphs have been obtained successively [3].

The notion of adjoint polynomials of graphs was first introduced by Liu [5]. In the search for chromatically unique graphs, it turned out that many brand-new results could be obtained by using the adjoint polynomials of these graphs. In this paper, we shall describe this method and give some simpler proofs of relevant theorems. Finally, we shall obtain a new family of chromatically unique graphs.

For a graph G, let $V(G)$ denote the vertex set of G, and $E(G)$ the edge set of G, and let $\bar{G}$ denote the complement of G. We also denote by $G_1\cup G_2$ the union of two disjoint graphs G_1 and G_2, and by nG the union of n disjoint copies of G.

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2. Basic properties of adjoint polynomials

Let G be a graph with p vertices. If G_0 is a spanning subgraph of G and every component of G_0 is complete, then G_0 is called an ideal subgraph of G. Let $b_i(G)$ denote the number of ideal subgraphs with $p-i$ components where $0 \leqslant i \leqslant p-1$. By Theorem 15 in [14], we can easily obtain the following formula

$$P(\bar{G},\lambda) = \sum_{i=0}^{p-1} b_i(G)(\lambda)_{p-i}, \tag{1}$$

where $p = |V(G)|$ and $(\lambda)_k = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-k+1)$.

Definition 1. If G is a graph with p vertices, then the polynomial

$$h(G,x) = \sum_{i=0}^{p-1} b_i(G)x^{p-i} \tag{2}$$

is called the adjoint polynomial of G.

A graph G is adjointly unique if $h(H,x) = h(G,x)$ implies that H is isomorphic to G. It is obvious that a graph G is adjointly unique if and only if $\bar{G}$ is chromatically unique.

Remark. Korfhage [4] showed that the σ-polynomial of a graph G can be written as

$$\sigma(G) = \sum_{i=0}^{k} a_i \sigma^{k-i},$$

where $k = |V(G)| - \chi(G), \chi(G)$ is the chromatic number of G and a_i is the number of ways of partitioning $V(G)$ into $|V(G)| - i$ disjoint independent sets. It is easy to show that

$$h(G,\sigma) = \sigma^{\chi(\bar{G})} \cdot \sigma(\bar{G}).$$

Since there is no an obvious relation between $\sigma(G)$ and $|V(G)|$, it is not very convenient to prove the chromatic uniqueness of graphs by using σ-polynomials. Let C_p denote the cycle with p vertices. We know that $\bar{C}_5$ is chromatically unique and $\overline{C_5 \cup K_1}$ is not chromatically unique, but they have the same σ-polynomial $\sigma^2 + 5\sigma + 5$. However, the graphs C_5 and $C_5 \cup K_1$ have different adjoint polynomials: $x^5 + 5x^4 + 5x^3$ and $x^6 + 5x^5 + 5x^4$, respectively.

Now we give some basic properties of adjoint polynomials.

Theorem 1. *Let G be a graph with k components $G_1, G_2, \ldots, G_k$; then* $h(G,x) = h(G_1,x)h(G_2,x)\cdots h(G_k,x)$.

Theorem 1, in fact, is a rewrite of Theorem 4 in [14].

Theorem 2 (Liu [6]). *Let G be a graph with $p(G)$ vertices, $q(G)$ edges, $A(G)$ triangles and degree sequence $(d_1, d_2, \ldots, d_p)$; then*

(i) $b_0(G) = 1$;

(ii) $b_1(G) = q(G)$;

(iii) $b_2(G) = A(G) + \binom{q(G)+1}{2} - 1/2\sum_{i=1}^{p} d_i^2$.

In order to simplify the proofs of the following theorems, we introduce an index $R(G)$ of graph G.

Definition 2. Let $f(x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n$ be a polynomial of x where $b_1, b_2, \ldots, b_n$ are nonnegative integers; we define

$$R(f) = \begin{cases} 0 & \text{if } b_1 = 0 \\ b_2 - \binom{b_1 - 1}{2} + 1 & \text{if } b_1 > 0 \end{cases}$$

and $R(G) = R(h(G,x))$ where G is a graph and $h(G,x)$ is its adjoint polynomial.

Liu and Chen [10], in fact, proved the following theorem.

Theorem 3. *Let $G_1, G_2, \ldots, G_k$ be the components of G; then*

$$R(G) = \sum_{i=1}^{k} R(G_i).$$

Theorem 4. *Let G be a connected graph such that $q(G) \geqslant 2$. If $e \in E(G)$, then $R(G) \leqslant R(G-e)$.*

Proof. Let $q = q(G), e = ab, N_G(a) = \{u \mid ua \in E(G)\}, N_G(b) = \{v \mid vb \in E(G)\}, d_G(a) = |N_G(a)|$, and $d_G(b) = |N_G(b)|$. We assume that $|N_G(a) \cap N_G(b)| = k, d_G(a) = k + \delta_1 + 1$, and $d_G(b) = k + \delta_2 + 1$. Then we have

$$\begin{aligned} b_2(G-e) &= A(G-e) + \binom{q}{2} - \frac{1}{2}\left[\sum_{\substack{x \in V(G) \\ x \neq a,b}} d_G^2(x) + (d_G(a)-1)^2 + (d_G(b)-1)^2\right] \\ &= A(G) - k + \binom{q+1}{2} - q - \frac{1}{2}\sum_{x \in V(G)} d_G^2(x) + d_G(a) + d_G(b) - 1 \\ &= b_2(G) + (k + \delta_1 + \delta_2 + 1) - q. \end{aligned}$$

Thus,

$$R(G-e)-R(G)=\left[b_2(G-e)-\binom{q-2}{2}+1\right]-\left[b_2(G)-\binom{q-1}{2}+1\right]$$
$$=b_2(G)+(k+\delta_1+\delta_2+1)-q-\binom{q-2}{2}-b_2(G)+\binom{q-1}{2}$$
$$=k+\delta_1+\delta_2-1.$$

Since G is connected and $q(G)\geqslant 2$, we know that $k+\delta_1+\delta_2\geqslant 1$, and hence $R(G)\leqslant R(G-e)$. □

Let P_n be the path with n vertices. We denote by D_n the graph consisting of K_3 and P_{n-2} by coinciding a vertex of K_3 with a vertex of degree 1 of P_{n-1}. Let F_n $(n\geqslant 6)$ be the graph obtained from D_{n-2} and K_3 by coinciding the vertex of degree 1 of D_{n-2} with a vertex of K_3. Finally, let T_n be a tree with three vertices of degree 1, $n-4$ vertices of degree 2 and one vertex of degree 3.

Theorem 5. *Let G be a nontrivial graph with n vertices. Then*
(i) $R(G)\leqslant 1$, *and the equality holds if and only if* $G\cong P_n(n\geqslant 2)$ *or* $G\cong K_3$;
(ii) $R(G)=0$, *if and only if G is one of the graphs C_n, D_n and T_n where* $n\geqslant 4$;
(iii) $R(G)=-1$ *and* $q(G)\geqslant p(G)+1$, *if and only if* $G\cong F_n(n\geqslant 6)$ *or* $G\cong K_4-e$.

Proof. It is easy to prove the sufficiency. Now we prove the necessity.

Let p, q, A and R represent briefly $p(G)$, $q(G)$, $A(G)$ and $R(G)$, respectively. Let $q=p+\beta(\beta\geqslant -1)$; then

$$R=b_2-\binom{q-1}{2}+1$$
$$=A+\binom{q+1}{2}-\frac{1}{2}\sum_{i=1}^{p}d_i^2-\binom{q-1}{2}+1,$$

and hence

$$\sum_{i=1}^{p}d_i^2=4q+2A-2R.$$

Since $2q=\sum_{i=1}^{p}d_i$, we have $\sum_{i=1}^{p}d_i^2=2\sum_{i=1}^{p}d_i+2A-2R$ and thus

$$\sum_{i=1}^{p}d_i(d_i-2)=2A-2R. \tag{3}$$

Since $\sum_{i=1}^{p}d_i=2p+2\beta$, it follows that

$$\sum_{i=1}^{p}(d_i-2)=2\beta. \tag{4}$$

From (3) and (4), we have

$$\sum_{i=1}^{p}(d_i - 1)(d_i - 2) = 2A - 2R - 2\beta. \tag{5}$$

Notice that every vertex of degree 1, degree 2, degree 3 and degree 4 of G contributes 0, 0, 2 and 6 to the sum

$$\sum_{i=1}^{p}(d_i - 1)(d_i - 2),$$

respectively, and we always have that $\sum_{i=1}^{p}(d_i - 1)(d_i - 2) \geqslant 0$.

Let $\Delta(G)$ denote the maximum degree of G. The following cases are to be considered:

Case 1: $\beta = -1$. In this case, G is a tree and hence $A = 0$.

Case 1.1: $R = 1$. From (5), we know that $\sum(d_i - 1)(d_i - 2) = 0$ (from here on, $\sum$ means $\sum_{i=1}^{p}$). This implies that $\Delta(G) = 2$, and hence $G \cong P_n$.

Case 1.2: $R = 0$. From (5), it follows that $\sum(d_i - 1)(d_i - 2) = 2$. Thus, G has exactly one vertex of degree 3 and $n - 1$ vertices of degree 1 or 2. Therefore, $G \cong T_n$.

Case 2: $\beta = 0$. In this case, G has exactly one cycle, and hence $0 \leqslant A \leqslant 1$.

Case 2.1: $R = 1$. From (5), it follows that $\sum(d_i - 1)(d_i - 2) = 2A - 2$, and $2A - 2 \geqslant 0$ implies that $A \geqslant 1$. Thus, $A = 1$ and $\sum(d_i - 1)(d_i - 2) = 0$. Therefore, $\Delta(G) = 2$ and hence $G \cong K_3$.

Case 2.2: $R = 0$. From (5), it follows that $\sum(d_i - 1)(d_i - 2) = 2A$. If $A = 0$, then $\sum(d_i - 1)(d_i - 2) = 0$. Thus, $\Delta(G) = 2$ and $G \cong C_n$ where $n \geqslant 4$. If $A = 1$, then $\sum(d_i - 1)(d_i - 2) = 2$. Thus, G has one vertex of degree 3 and $n - 1$ vertices of degree 1 or 2, hence $G \cong D_n$ where $n \geqslant 4$.

Case 3. $\beta = 1$. G has exactly two cycles. Since G is connected, G has at least two vertices of degree 3 or $\Delta(G) = 4$. Thus, we have

$$\sum(d_i - 1)(d_i - 2) \geqslant 4. \tag{6}$$

Case 3.1. $R = 1$. From (5), it follows that $\sum(d_i - 1)(d_i - 2) = 2A - 4$. From (6), it follows that $2A - 4 \geqslant 4$ and $A \geqslant 4$. This is a contradiction, since G has exactly two cycles. Namely, there exists no graph such that $\beta = 1$ and $R = 1$. In Table 1, we use the word 'nonexistence' to express such a case.

Case 3.2. $R = 0$. From (5), it follows that $\sum(d_i - 1)(d_i - 2) = 2A - 2$. By (6), we know that $2A - 2 \geqslant 4$ and $A \geqslant 3$, and this is a contradiction.

Case 3.3. $R = -1$. From (5), it follows that $\sum(d_i - 1)(d_i - 2) = 2A$. By (6), we have $A \geqslant 2$, and hence $A = 2$. Thus, $\sum(d_i - 1)(d_i - 2) = 4$, and hence G has exactly two vertices of degree 3 and $n - 2$ vertices of degree 2. Therefore, $G \cong F_n$ or $G \cong K_4 - e$,

Case 4. $\beta = 2, R = -1$. Since $q = p + 2$, we have $\sum(d_i - 2) = 4$ and

$$\sum(d_i - 1)(d_i - 2) \geqslant 8. \tag{7}$$

Table 1

β G R	-1	0	1	2
1	P_n $(n \geqslant 2)$	K_3	Nonexistence	
0	T_n $(n \geqslant 4)$	C_n, D_n $(n \geqslant 4)$	Nonexistence	
-1			F_n $(n \geqslant 6)$, $K_4 - e$	Nonexistence

From (5), it follows that $\sum (d_i - 1)(d_i - 2) = 2A - 2$, and by (7), we know that $2A - 2 \geqslant 8$ and $A \geqslant 5$. This is a contradiction.

From the above discussion, we get Table 1.

It is not difficult to prove the following proposition: if $d_1, d_2, \ldots, d_p$ are positive integers, such that $\sum_{i=1}^{p} d_i = 2p - 2$, then $\sum_{i=1}^{p} d_i^2$ has its maximum if and only if $|d_i - d_j| \leqslant 1$ where $1 \leqslant i, j \leqslant p$. In other words,

$$\max\{R(T) \mid T \text{ is a tree with } n \text{ vertices}\} \leqslant R(P_n) = 1.$$

Let G be a graph with n vertices and let T be its spanning tree, then, by Theorem 4, $R(G) \leqslant R(T) \leqslant 1$. Also from Theorem 4, it is easy to see that if there exists no graph G such that $R(G) \geqslant k$ and $q(G) = p(G) + \beta$ then no graph G such that $R(G) = k$ and $q(G) = p(G) + \beta + 1$. In other words, in Table 1, if a block has been written as 'nonexistence', then all the right blocks in the same row must be written as 'nonexistence', and the theorem is proved. □

Remark. Parts (1) and (2) of Theorem 5 were proved in [6,9,7] by a more complex method; the above proof is a generalized and simpler proof.

Definition 3. Let $h(G,x)$ be a polynomial and

$$h(G,x) = x^{\alpha(G)} h_1(G,x),$$

where $x^{\alpha(G)}$ is the minimal power of x with nonzero coefficient in $h(G,x)$. If $h_1(G,x)$ is an irreducible polynomial over the rational number field, then G is called an irreducible graph.

By the Unique Factorization Theorem of polynomials over the rational number field and some of the basic properties of adjoint polynomials, the papers [6,8–12] obtained some families of adjointly unique graphs, whose complement graphs are chromatically unique. Using Theorems 3 and 5, we can give shorter and simpler proofs for the main theorems of the mentioned papers. Here is an example.

Theorem 6 (Liu [6], and Liu and Li [11]). *If $G = P_{s_1} \cup P_{s_2} \cup \cdots \cup P_{s_r} \cup mK_1$, and P_{s_i} is an irreducible path for $1 \leqslant i \leqslant r$ and $s_i \neq 4$, then $\bar{G}$ is chromatically unique graph.*

Proof. It will suffice to prove that G is adjointly unique graph. Suppose that

$$h(H,x) = h(G,x) \tag{8}$$

and $H = nK_1 \cup H_1 \cup \cdots \cup H_t$, where $H_i \ncong K_1$ for $1 \leqslant i \leqslant t$. By (8) and Theorem 1, we have

$$x^n \prod_{i=1}^{t} h(H_i,x) = x^{\alpha(G)} \prod_{i=1}^{r} h_1(P_{s_i},x), \tag{9}$$

where $\alpha(G) = m + \sum_{i=1}^{r} \alpha(P_{s_i})$. Also suppose that the right-hand side of (9) has been uniquely factored into irreducible polynomials over the rational number field. Thus, there exists $B \subseteq \{1,2,\ldots,r\}$ such that

$$h(H_i,x) = x^{\alpha} \prod_{i \in B} h_1(P_{s_i},x) \tag{10}$$

Let $g(x)$ denote the right polynomial of (10). Since $R(h(G)) = R(h_1(G))$, by Theorem 3, we know that $R(g) = |B|$, and hence $R(H_1) = |B|$. Because H_1 is a nontrivial connected graph, by Theorem 5, we have

$$|B| = R(H_1) \leqslant 1.$$

If $|B| = 0$, then (10) does not hold, and thus we know that $|B| = 1$. Without loss of generality, we can assume that $B = \{1\}$. From $R(H_1) = 1$, we know that $H_1 \cong P_n$ or $H_1 \cong K_3$. But $H_1 \ncong K_3$ because $q(H_1) = q(P_{s_1}) \neq 3$. Thus $H_1 \cong P_{s_1}$. Similarly, we can prove that H_i $(i = 2,3,\ldots,t)$ is a path and isomorphic to a nontrivial component of G. From (9), we know that $t = r$, $n = m$ and thus $H \cong G$. $\square$

In a similar way, the following results in [8–10,12] can be proved: if C_{m_i}, P_{n_j} and D_{m_i} are irreducible, then $\bigcup_i C_{m_i}, \bigcup_i C_{m_i} \bigcup_j P_{n_j}, \bigcup_i D_{m_i}, \bigcup_i C_{m_i} \bigcup_j D_{n_j}$ and $\bigcup_i D_{m_i} \bigcup_j P_{n_j}$ are adjointly unique. In [8–10,12], the irreducibility of some C_m, P_n and D_m was proved.

3. A new family of chromatically unique graphs

Theorem 7. *If F_n is irreducible for $n \geqslant 6$, then $\bar{F}_n$ is chromatically unique.*

Proof. It will suffice to prove that F_n is adjointly unique. Let

$$h(H,x) = h(F_n,x) \tag{11}$$

and $H = rK_1 \cup H_1 \cup \cdots \cup H_s$, where $H_i \ncong K_1, i = 1,2,\ldots,s$. By (11), we have

$$x^r h(H_1,x) \cdot h(H_2,x) \ldots h(H_s,x) = x^{\alpha(F_n)} h_1(F_n,x). \tag{12}$$

Since the right polynomial of (12) has only one irreducible factor $h_1(F_n,x)$ of degree d where $d \geqslant 2$, by the Unique Factorization Theorem of polynomial over the rational number field, H has exactly one nontrivial component. This implies that $s = 1$ and

$$h(H_1,x) = x^t h_1(F_n,x).$$

Thus, the sequence of coefficients of $h(H_1,x)$ is the same as of $h_1(F_n,x)$, and hence as of $h(F_n,x)$. From this fact, we know that $R(H_1) = R(F_n) = -1$, $q(H_1) = q(F_n) = n+1$, and then $p(H_1) \leqslant p(H) = p(F_n) = n = q(H_1) - 1$. By (iii) of Theorem 5, it follows that $H_1 \cong F_n$, and hence $p(H_1) = n$ and $r = 0$; in other words,

$$H = H_1 \cong F_n.$$

Therefore, F_n is adjointly unique, and hence $\bar{F}_n$ is chromatically unique. □

Now we discuss the irreducibility of F_n.

Lemma 1. *Let $e = uv$ be an edge of G. If e is not an edge of any triangle of G, then*

$$h(G,x) = h(G - uv,x) + xh(G - \{u,v\},x).$$

Lemma 1 is a rewrite of Theorem 1 in [14]. By Lemma 1, it is easy to prove the following lemmas.

Lemma 2. $h(D_{n+2},x) = x(h(D_n,x) + h(D_{n+1},x)) \quad (n \geqslant 3).$

Theorem 8. $h(F_{n+2},x) = x(h(F_n,x) + h(F_{n+1},x)) \quad (n \geqslant 6).$

Proof. By straightforward computation, we know that the theorem holds for $6 \leqslant n \leqslant 9$. For $n \geqslant 10$, by Lemmas 1 and 2, we have

$$\begin{aligned} h(F_{n+2}) &= h(D_6 \cup D_{n-4}) + xh(D_5 \cup D_{n-5}) \\ &= h(D_6)h(D_{n-4}) + xh(D_5)h(D_{n-5}) \\ &= x(h(D_4) + h(D_5))h(D_{n-4}) + x^2(h(D_3) + h(D_4))h(D_{n-5}) \\ &= x(h(D_4)h(D_{n-4}) + xh(D_3)h(D_{n-5}) + h(D_5)h(D_{n-4}) + xh(D_4)h(D_{n-5})) \\ &= x(h(F_n) + h(F_{n+1})). \quad \square \end{aligned}$$

The recursive formula is given in Theorem 8. By this theorem and the theorems of factoring rational and integral polynomials, we know that $h_1(F_6,x)$ and $h_1(F_{10},x)$ are irreducible. In fact,

$$\begin{aligned} h_1(F_6,x) &= x^4 + 7x^3 + 13x^2 + 7x + 1, \\ h_1(F_{10},x) &= x^6 + 11x^5 + 43x^4 + 72x^3 + 51x^2 + 14x + 1. \end{aligned}$$

From [13], we know that these two polynomials are irreducible over the Galois field GF(2), and hence irreducible over the rational number field. Therefore, $\bar{F}_6$ and $\bar{G}_{10}$ are chromatically unique.

We end this paper by proposing the following conjecture.

Conjecture. There exists an infinite number of irreducible F_n.

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